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Abstract

In [9], a model of a private ownership economy is presented in which production and consumption bundles are treated separately. Each of the two types of bundles is assumed to establish a convex cone. The main part in the modelling is the introduction of production technologies which can be thought of as replacing the notion of production sets in Arrow and Debreu's model. In this paper, it is proved that under mild economically interpretable conditions, presented in [9], a Walrasian equilibrium exists.

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1 Introduction

In [9], a new mathematical model of a private ownership economy, a corresponding Walrasian equilibrium theorem and the mathematics incorporated, are presented. Apart from this introduction, this paper is devoted solely to the proof of the Existence Theorem presented in the afore mentioned paper.

The model in [9] differs from the neo-classical models, described in the standard works of [4] and [1], in the following two features.

- The model recognises production and consumption as two different economic features. Thus, two different types of economy bundles occur: production bundles and consumption bundles. Bundles of both types can be consumed by economic agents and bundles of both types will be present in the initial endowment. However, the production processes can convert only production bundles into consumption bundles and not the other way around.

- Also, the idea of [7] is followed. In [7] a mathematical model of a pure exchange economy is presented in which commodities are not assumed to occur separately. Instead of introducing the commodity space $(\mathbb{R}^n)^+$ describing n different commodities, only appearance of so called economy bundles is assumed.

The model of a private ownership economy, presented in [9], is only in terms of convex cones and their properties, and not in terms of vector spaces, whereas the neo-classical models are set in terms of a finite-dimensional Euclidean space. The use of convex cones is emphasized by the axiomatic introduction of the concept of salient half-space. For every salient half-space C, the vector space generated by C is denoted by $V[C]$, and the partial order relation induced by C is denoted by $\geq_C$. In case $V[C]$ is finite-dimensional, the unique linear topology on $V[C]$, induced by any chosen norm on $V[C]$ is denoted by $\mathcal{T}$. In [9], a non-vector-space-related description of the relative topology of $\mathcal{T}$ on C is presented. Since C is total in $V[C]$, the set $\mathrm{int}(C)$, consisting of all internal points, is non-empty.

Using the general concept of salient half-space in our model, we do not introduce the concept of a commodity but consider the concept of "economy bundle", which carries the characteristics of exchangable objects in the economy, instead. In a worldlike example, our model can describe the non-neo-classical situation in which fixed links between different commodities are present, for instance an economy in which only fixed, prescribed combinations of commodities can be traded. Examples are special pre-packed offers, or free (sample)-products received when purchasing a commodity. Also, this model can describe a situation in which the preferences of the agents are in terms of characteristics of commodities instead of in terms of the commodities themselves. In the labour market, for instance, a firm may ask for an employee with a certain education, intelligence and working experience. In this setting, one can consider an "economy bundle" to be a person with such (and perhaps other) specific attributes. In general, an "economy bundle" can be considered to be a carrier of several attributes (cf. the work of Lancaster, [6]). Moreover, the same attribute may appear in more than one economy bundle. This mixture of attributes can be inextricable both in characteristics and in time.

An economy bundle is assumed to be a unique concatenation of a production (economy) bundle and a consumption (economy) bundle. Here, only production bundles can be used as input for a production process whereas the output of this process is always a consumption bundle. The set C of economy bundles is taken to be the product set $C_{\text{prod}} \times C_{\text{cons}}$ where the salient half-spaces C_{prod} and C_{cons} contain the production and consumption bundles, respectively. Both C_{prod} and C_{cons} are assumed to be non-trivial, i.e., assumed to be not equal to $\{0^{\text{prod}}\}$ and $\{0^{\text{cons}}\}$, where 0^{prod} and 0^{cons} denote the vertex of C_{prod} and C_{cons}, respectively. As a consequence, C is also non-trivial. In every economy bundle $x \in C$, each of the two types is uniquely represented: $x = (x^{\text{prod}}, x^{\text{cons}})$ with $x^{\text{prod}} \in C_{\text{prod}}$ and $x^{\text{cons}} \in C_{\text{cons}}$.

Each economy bundle $x \in C$ represents a production process in which consumption bundle $x^{\text{cons}} \in C_{\text{cons}}$ is considered to be obtained as output from production bundle $x^{\text{prod}} \in C_{\text{prod}}$ as input. A collection $T \subset C$ of production processes is called a production technology if

a) $(0^{\text{prod}}, 0^{\text{cons}}) \in T$,

b) If $(0^{\text{prod}}, x^{\text{cons}}) \in T$ then $x^{\text{cons}} = 0^{\text{cons}}$,

c) $T = \bigcup_{e \in E(T)} F_e$.

Here, $F_e = \{x \in C \mid e^{\text{prod}} \leq_{\text{prod}} x^{\text{prod}} \text{ and } x^{\text{cons}} \leq_{\text{cons}} e^{\text{cons}}\}$, and $E(T)$ denotes the set of all efficient production processes in T. A production process $(x^{\text{prod}}, x^{\text{cons}})$ of a technology T is called efficient, if at least x^{prod} is needed to produce x^{cons}, and if it is not possible to produce more than x^{cons} out of x^{prod}, i.e., for a production technology T, a production process $e \in T$ is efficient if $\forall x \in C$:

- $((x^{\text{prod}}, e^{\text{cons}}) \in T \text{ and } x^{\text{prod}} \leq_{\text{prod}} e^{\text{prod}}) \Longrightarrow x^{\text{prod}} = e^{\text{prod}}$;
- $((e^{\text{prod}}, x^{\text{cons}}) \in T \text{ and } e^{\text{cons}} \leq_{\text{cons}} x^{\text{cons}}) \Longrightarrow e^{\text{cons}} = x^{\text{cons}}$.

In the presented model, there are J production technologies, indexed by $j \in \{1, \ldots, J\}$.

As mentioned above, commodities are not assumed to occur separately. Hence, the price of a single commodity is not a meaningful concept. Instead, the value of an economy bundle is introduced. This value is determined on the basis of "pricing functions", which are described by subadditive positive functionals on C. The set of all such functionals has been introduced in [9] as the salient half-dual space C^* and we have seen that $C^* = (C_{\text{prod}})^* \oplus (C_{\text{cons}})^*$. Let $x \in C$ and $p \in C^*$, then the value $\mathcal{V}(x,p)$ of economy bundle x with respect to the pricing function p equals

$$
\begin{aligned}
\mathcal{V}(x,p) &:= p^{\text{prod}}(x^{\text{prod}}) + p^{\text{cons}}(x^{\text{cons}}) \\
&= [x,p]_C = [x^{\text{prod}}, p^{\text{prod}}]_{\text{prod}} + [x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}}.
\end{aligned}
$$

Given a pricing function $p \in C^*$ and a production process $x \in C$, the gain $\mathcal{G}(x,p)$ of the pair (x,p) equals

$$\mathcal{G}(x,p) := [x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} - [x^{\text{prod}}, p^{\text{prod}}]_{\text{prod}}.$$

Given $j \in \{1, \ldots, J\}$ and $p \in C^*$, the (possibly empty) set of all gain maximizing production processes in production technology T_j is called the supply set $S_j(p)$, i.e.,

$$S_j(p) = \{x \in T_j \mid \forall y \in T_j : \mathcal{G}(x,p) \geq \mathcal{G}(y,p)\}.$$

The conditions on each T_j and the definition of $E(T_j)$ imply that $\forall j \in \{1, \ldots, J\}$ $\forall p \in \mathrm{int}(C^*) : S_j(p) \subseteq E(T_j)$.
For $j \in \{1, \ldots, J\}$, let Domain[j] be defined by

$$\mathrm{Domain}[j] := \{p \in \mathrm{int}(C^*) \mid S_j(p) \neq \emptyset\}.$$

Furthermore, let the set Domain $\subset \mathrm{int}(C^*)$ be defined by

$$\mathrm{Domain} := \bigcap_{j=1}^{J} \mathrm{Domain}[j].$$

One of the assumptions that will be made is Domain $\neq \emptyset$.
For given $j \in \{1, \ldots, J\}$ and $p \in$ Domain[j] we denote the maximum gain of production technology T_j by $\mathcal{G}_j(p)$. Note that $\forall x \in S_j(p) : \mathcal{G}(x,p) = \mathcal{G}_j(p)$.

The features of an economic agent are an economy bundle $w = (w^{\mathrm{prod}}, w^{\mathrm{cons}}) \in C$, called initial endowment, a preference relation $\succeq$ defined on C, and share rates in the gain of the production technologies. There are I agents in the economy, indexed by $i \in \{1, \ldots, I\}$. For each $i \in \{1, \ldots, I\}$, the share rates θ_{ij}, $j \in \{1, \ldots, J\}$, satisfy $\theta_{ij} \in [0, 1]$ and $\sum_{i=1}^{I} \theta_{ij} = 1$.
At pricing function $p \in$ Domain the income $\mathcal{K}_i(p)$ of agent i is defined by

$$\mathcal{K}_i(p) := \mathcal{V}(w_i, p) + \sum_{j=1}^{J} \theta_{ij} \mathcal{G}_j(p),$$

where the first term denotes the value of the initial endowment of agent i and the second term denotes the total value received from shares in the gain of the production technologies. For a given pricing function $p \in$ Domain, the budget set $B_i(p) := \{x \in C \mid \mathcal{V}(x,p) \leq \mathcal{K}_i(p)\}$ consists of all economy bundles that can be afforded given pricing function p and value $\mathcal{K}_i(p)$. The set $D_i(p) := \{x \in B_i(p) \mid \forall y \in B_i(p) : x \succeq y\}$ of all best (most preferable) elements of the budget set $B_i(p)$, is called the demand set of agent i at pricing functional $p \in$ Domain.
In this setting, an equilibrium concept analogous to that of the neo-classical Walrasian equilibrium can be introduced.

Definition 1.1 *A Walrasian equilibrium is an* $(J+I+1)$*-tuple* $((s_j)_{j=1}^{J}, (d_i)_{i=1}^{I}, p_{eq})$ *consisting of*

- $p_{eq} \in C^* \setminus \{0\}$,
- $s_j \in S_j(p_{eq})$ *for all* $j \in \{1, \ldots, J\}$;
- $d_i \in D_i(p_{eq})$ *for all* $i \in \{1, \ldots, I\}$;
- $\sum_{i=1}^{I} d_i + \sum_{j=1}^{J} ({s_j}^{\mathrm{prod}}, 0^{\mathrm{cons}}) = \sum_{i=1}^{I} w_i + \sum_{j=1}^{J} (0^{\mathrm{prod}}, {s_j}^{\mathrm{cons}})$.

We call p_{eq} a (Walrasian) equilibrium pricing function.

We end this introduction by recalling the Equilibrium Existence Theorem of [9] in which the existence of an equilibrium pricing function is guaranteed. Furthermore, we discuss shortly the mathematical conditions stated in this theorem.

Equilibrium Existence Theorem

The model of a private ownership economy, described above, admits a Walrasian equilibrium, under the following assumptions:

Assumption 1 $V[C]$ *is finite-dimensional.*

Assumption 2 $C^{**} = C$.

Assumption 3 *For every* $j \in \{1, \ldots, J\}$, *production technology* T_j *satisfies*

- **a)** *if* $e_1, e_2 \in E(T_j)$, $e_1 \neq e_2$, $\tau \in (0,1)$ *then* $\tau e_1 + (1-\tau)e_2 \in T_j$ *and* $\tau e_1 + (1-\tau)e_2 \notin E(T_j)$,
- **b)** T_j *is closed with respect to topology* $\mathcal{T}(C, C^*)$.

Assumption 4 *For every* $i \in \{1, \ldots, I\}$, *preference relation* $\succeq_i$ *is*

- **a)** *monotone:* $\forall\, x, y \in C : x \leq_C y$ *implies* $y \succeq_i x$,
- **b)** *strictly convex:* $\forall x, y \in C$, $\tau \in (0,1) : x \succeq_i y$ *and* $x \neq y$ *imply* $\tau x + (1-\tau)y \succ_i y$,
- **c)** *continuous:* $\forall y \in C$ *the sets* $\{x \in C \mid x \succeq_i y\}$ *and* $\{x \in C \mid y \succeq_i x\}$ *are closed in* C.

Assumption 5

- **a)** $\exists p \in \mathrm{int}(C^*)\ \forall j \in \{1, \ldots, J\} : S_j(p) \neq \emptyset$,
- **b)** *for every sequence* $(p_n)_{n \in I\!N}$ *in Domain with limit* $p \in \partial C^* \setminus \{0\}$, *there is* $i_0 \in \{1, \ldots, I\}$ *such that* $\liminf_{n \to \infty} \{\mathcal{K}_{i_0}(p_n) \mid n \in I\!N\} > 0$.

Assumptions 1 and 2 guarantee that C is a closed subset of $V[C]$, with respect to topology $\mathcal{T}$. Furthermore, they guarantee that every bounded set in C is pre-compact and so the budget sets are compact for interior pricing functions. Assumptions 3.a and 3.b imply that instead of dealing with supply sets, we deal with supply functions. In order to guarantee that supply is unique, Assumption 3.a is introduced, which resembles “decreasing returns to scale” or “strictly convex production sets”. Assumption 3.b guarantees the continuity of the supply functions. Similarly, Assumption 4 implies that we can deal with continuous demand functions. All this will be shown in the appendix. Assumption 5.a yields that the total supply function has a non-trivial domain. Existence of a Walrasian equilibrium, in the sense of Definition 1.1, follows from a generalisation of Brouwers' Fixed Point Theorem for continuous functions on salient half-spaces (cf. Proposition 2.7). In this, Assumption 5.b will be used. In [9], two economically interpretable conditions are introduced which, together, imply the less transparent, but weaker Assumption 5.b.

Finally, we mention that throughout this paper small letters (x, y, z, p, q) are used to denote elements of the salient half-spaces C and C^*, capital letters (S, B, D) denote subsets of C and C^*, greek letters (θ, λ, τ) denote scalars, and capital script letters $(\mathcal{S}, \mathcal{D}, \mathcal{F})$ denote functions.

2 Existence of equilibrium

In this section, we shall prove the Equilibrium Existence Theorem concerning the model presented in [9] and shortly summarised in the introduction.

Preliminaries

At the end of the previous section we stated that as a consequence of Assumption 3 and Assumption 4, we can deal with continuous supply and demand functions instead of supply and demand sets. In the following three lemmas, the consequences of Assumptions 3 and 4 are described in more detail. The proof of these lemmas can be found in the appendix.

Lemma 2.1 *Assumption 3 implies the following.*

1. *For all $j \in \{1, \dots, J\}$ and for all $p \in Domain[j]$, the supply set $S_j(p)$ consists of exactly one element.*
2. *For all $j \in \{1, \dots, J\}$, define the supply function $\mathcal{S}_j : Domain[j] \to E(T_j)$ such that $S_j(p) = \{\mathcal{S}_j(p)\}$, for all $p \in Domain[j]$. Then $\mathcal{S}_j$ is continuous on its domain.*
3. *For all $j \in \{1, \dots, J\}$ and for all $p_0 \in \mathrm{int}(C^*)$: if $(p_n)_{n \in I\!N}$ is a sequence in $Domain[j]$, convergent to $p \in \mathrm{int}(C^*) \setminus Domain[j]$, then $\limsup_{n \to \infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$.*

From the supply functions $\mathcal{S}_j$, $j \in \{1, \dots, J\}$, we define the total supply function $\mathcal{S}$: Domain $\to C$, as follows,

$$\forall p \in \mathrm{Domain} : \mathcal{S}(p) := \sum_{j=1}^{J} \mathcal{S}_j(p).$$

Note that $\mathcal{S}$ is continuous on its domain. Furthermore, since $\mathcal{S}(\lambda p) = \mathcal{S}(p)$ for all $\lambda > 0$ and $p \in$ Domain, the set Domain $\cup \{0\}$ is a, not necessarily convex, subcone of C^*, i.e., $\forall p \in \mathrm{Domain} \cup \{0\}\ \forall \alpha \geq 0 : \alpha p \in \mathrm{Domain} \cup \{0\}$. Also, note that Assumption 5.a guarantees Domain $\neq \emptyset$.

Corollary 2.2 *If $(p_n)_{n \in I\!N}$ is a sequence in Domain, convergent to $p \in \mathrm{int}(C^*) \setminus Domain$, then $\limsup_{n \to \infty} \mathcal{G}(\mathcal{S}(p_n), p_0) = -\infty$ for any $p_0 \in \mathrm{int}(C^*)$.*

Proof
Let $p_0 \in \mathrm{int}(C^*)$. For al $j \in \{1, \dots, J\}$ either $p \in$ Domain$[j]$ and $\limsup_{n \to \infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0)$ is finite (Lemma 2.1.(2)), or $p \notin$ Domain$[j]$ and $\limsup_{n \to \infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$ (Lemma 2.1.(3)). Since $\exists j_0 \in \{1, \dots, J\} : p \notin$ Domain$[j_0]$, we conclude $\limsup_{n \to \infty} \mathcal{G}(\mathcal{S}(p_n), p_0) = -\infty$. □

Lemma 2.3 *Assumption 4 implies the following.*

1. *For all $i \in \{1, \dots, I\}$ and for all $p \in Domain$, the demand set $D_i(p)$ consists of exactly one element.*

2. *For all $i \in \{1,\ldots,I\}$, define the demand function $\mathcal{D}_i$: Domain $\to C$, such that $D_i(p) = \{\mathcal{D}_i(p)\}$, for all $p \in$ Domain. Then $\mathcal{D}_i$ is continuous on its domain.*

3. *For all $i \in \{1,\ldots,I\}$: if $(p_n)_{n\in I\!N}$ is a sequence in Domain, convergent to $p \in \partial C^*$, and if the sequence $(\mathcal{K}_i(p_n))_{n\in I\!N}$ is convergent with limit $\kappa > 0$ then the sequence $(\mathcal{D}_i(p_n))_{n\in I\!N}$ is unbounded.*

Analogously to the definition of the total supply function $\mathcal{S}$, we now define the total demand function $\mathcal{D}$: Domain $\to C$, as

$$\forall p \in \text{Domain} : \mathcal{D}(p) := \sum_{i=1}^{I} \mathcal{D}_i(p).$$

By Lemma 2.3, the total demand function is continuous. The following corollary is a direct result of Assumption 5.b and Lemma 2.3.(3).

Corollary 2.4 *If $(p_n)_{n\in I\!N}$ is a sequence in Domain, convergent to $p \in \partial C^* \setminus \{0\}$, then the sequence $(\mathcal{D}(p_n))_{n\in I\!N}$ is unbounded.*

Furthermore, Assumption 4 implies a version of Walras' Law, adapted to this model, the proof of which can also be found in the appendix.

Lemma 2.5 (Walras' law) *Let $p \in$ Domain. Then*

$$\mathcal{V}(\mathcal{D}(p), p) = \mathcal{V}(w_{\text{total}}, p) + \mathcal{G}(\mathcal{S}(p), p).$$

Next, for all $p \in$ Domain and $q \in C^*$, we introduce the notation

$$\mathcal{Z}(p,q) := \mathcal{V}(\mathcal{D}(p), q) - \mathcal{G}(\mathcal{S}(p), q) - \mathcal{V}(w_{\text{total}}, q).$$

The function $\mathcal{Z}$: Domain $\times C^* \to I\!R$ thus defined is bi-continuous; the adapted version of Walras' law (Lemma 2.5) reads

$$\forall p \in \text{Domain} : \mathcal{Z}(p,p) = 0. \tag{1}$$

Convenience of this notation is shown in the following characterisation of equilibrium pricing functions, which can be easily checked by the reader.

Lemma 2.6 *Let $p \in$ Domain. Then p is an equilibrium pricing function if and only if $\forall q \in C^* : \mathcal{Z}(p,q) \le 0$.*

In order to prove existence of equilibrium pricing functions, we construct an auxiliary function $\mathcal{H}$ on the salient half-space C^*, satisfying

- $\forall p \in C^* \setminus \{0\} : (\exists \alpha \ge 0 : \mathcal{H}(p) = \alpha p) \iff (p \in \text{Domain and } \forall q \in C^* : \mathcal{Z}(p,q) \le 0)$.
- $\mathcal{H}$ is continuous on $C^* \setminus \{0\}$.

Then the following generalisation of Brouwers' Fixed Point Theorem, proved in [9], can be used.

Proposition 2.7 *Let S be a salient half-space satisfying $V[S]$ is finite-dimensional and $S^{**} = S$. Let $\mathcal{F} : S \setminus \{0\} \to S$ be a continuous function, then there exists an $x \in S \setminus \{0\}$ such that $\mathcal{F}(x) = \alpha x$ for some $\alpha \ge 0$. In fact, for all $p_0 \in \text{int}(S^*)$ there is $x \in S$ such that $\mathcal{F}(x) = [\mathcal{F}(x), p_0]x$.*

By the above proposition, existence of a function $\mathcal{H}$ with the above mentioned properties implies existence of a Walrasian equilibrium. Hence, the remaining part of this section is dedicated to the construction of such a function on C^*.

Construction of an auxiliary function

In [9] it is shown that the section $L_1(x_0) := \{q \in C^* \mid \mathcal{V}(x_0, q) = 1\}$ is compact for every $x_0 \in \mathrm{int}(C)$. For the rather standard way of defining the Lebesgue measure μ on such a section, we also refer to [9].

Given some fixed $x_0 \in \mathrm{int}(C)$, the function $\mathcal{F}_0 : \mathrm{Domain} \to C^*$ is defined by

$$\mathcal{F}_0(p) := \int_{L_1(x_0)} \max\{0, \mathcal{Z}(p,q)\} q \, d\mu(q). \tag{2}$$

Note that for every $p \in$ Domain:

$$\mathcal{Z}(p, \mathcal{F}_0(p)) \geq 0. \tag{3}$$

We extend $\mathcal{F}_0$ to the whole of C^* as follows. From Assumption 5.a we conclude there exists $p_0 \in$ Domain. Now, the function $\mathcal{F} : C^* \to C^*$ is defined by

$$\mathcal{F}(p) := \begin{cases} (1 - \eta(\mathcal{Z}(p,p_0)))\mathcal{F}_0(p) + \eta(\mathcal{Z}(p,p_0))p_0 & p \in \mathrm{Domain} \\ p_0 & p \notin \mathrm{Domain}, \end{cases} \tag{4}$$

where η is the sigma-oidal function defined by

$$\eta(\phi) := \begin{cases} 0 & \text{if } \phi \leq 0 \\ \phi & \text{if } 0 < \phi < 1 \\ 1 & \text{if } 1 \leq \phi. \end{cases} \tag{5}$$

Note that

$$\forall \phi \in I\!R : \qquad \phi\eta(\phi) \geq 0, \text{ and} \tag{6}$$

$$\phi\eta(\phi) = 0 \iff \phi \leq 0. \tag{7}$$

Lemma 2.8 *Let $p \in C^*$. Then $(\exists\, \alpha \geq 0 : \mathcal{F}(p) = \alpha p) \iff (p \in \text{Domain and } \forall q \in C^* : \mathcal{Z}(p,q) \leq 0)$.*

Proof
Let $p \in$ Domain and $\forall q \in C^* : \mathcal{Z}(p,q) \leq 0$, then, by (2), $\mathcal{F}_0(p) = 0$ and by (5), $\eta(\mathcal{Z}(p,p_0)) = 0$. By (4), we conclude that $\mathcal{F}(p) = 0$.
For the converse, suppose $\mathcal{F}(p) = \alpha p$ for some $\alpha \geq 0$. From (4) and the fact that Domain$\cup \{0\}$ is a cone containing p_0, it follows that $p \in$ Domain. Walras' law (equation (1)) yields

$$\mathcal{Z}(p, \mathcal{F}(p)) = \alpha \mathcal{Z}(p,p) = 0.$$

By (4), (3) and (6) , we find

$$0 = \mathcal{Z}(p, \mathcal{F}(p)) = \underbrace{(1 - \eta(\mathcal{Z}(p,p_0)))\mathcal{Z}(p, \mathcal{F}_0(p))}_{\geq 0} + \underbrace{\eta(\mathcal{Z}(p,p_0))\mathcal{Z}(p,p_0)}_{\geq 0}.$$

Clearly,

$$(1 - \eta(\mathcal{Z}(p,p_0)))\mathcal{Z}(p, \mathcal{F}_0(p)) = 0 \tag{8}$$

and

$$\eta(\mathcal{Z}(p,p_0))\mathcal{Z}(p,p_0) = 0. \tag{9}$$

By (9) and (7) we find $\mathcal{Z}(p, p_0) \leq 0$, hence, using the definition of η, (8) implies

$$0 = \mathcal{Z}(p, \mathcal{F}_0(p)) = \int_{L_1(x_0)} \max\{0, \mathcal{Z}(p,q)\}\mathcal{Z}(p,q)d\mu(q).$$

So, for all $q \in L_1(x_0) : \mathcal{Z}(p,q) \leq 0$. □

Existence of $p \in$ Domain with $\mathcal{F}(p) = \alpha p$ will be proved by showing that the auxilary function $\mathcal{F}$ is continuous on $C^* \setminus \{0\}$, and then applying Proposition 2.7.
In order to prove that the auxiliary function $\mathcal{F}$ is continuous, we need the following lemma.

Lemma 2.9 *The function $\mathcal{F}_0$ is continuous on Domain.*

Proof
Recall the definition of x_0 and $L_1(x_0)$ in the definition of the auxiliary function $\mathcal{F}$. Impose on C^* the norm $\| \,.\, \|_{x_0}$, and let $\| \,.\, \|$ be the norm on C, dual to the norm $\| \,.\, \|_{x_0}$ (cf. [9]). Thus, by definition, for all $q \in L_1(x_0)$ we have $\| q \|_{x_0} = 1$.
And so, for all $p_1, p_2 \in$ Domain and $q \in C^*$:

$$\begin{aligned}
&| \mathcal{Z}(p_1, q) - \mathcal{Z}(p_2, q) | = \\
&|\mathcal{V}(\mathcal{D}(p_1), q) - \mathcal{G}(\mathcal{S}(p_1), q) - \mathcal{V}(\mathcal{D}(p_2), q) + \mathcal{G}(\mathcal{S}(p_2), q)| \leq \\
&\| \mathcal{D}(p_1) - \mathcal{D}(p_2) \| + \| \mathcal{S}^{\text{cons}}(p_1) - \mathcal{S}^{\text{cons}}(p_2) \| + \| \mathcal{S}^{\text{prod}}(p_1) - \mathcal{S}^{\text{prod}}(p_2) \| .
\end{aligned}$$

From this, and the fact that for all $\alpha, \beta \in I\!R : |\max\{0,\alpha\} - \max\{0,\beta\}| \leq |\alpha - \beta|$, we find

$$\begin{aligned}
&\| \mathcal{F}_0(p_1) - \mathcal{F}_0(p_2) \|_{x_0} \leq \\
&\int_{L_1(x_0)} | \max\{0, \mathcal{Z}(p_1, q)\} - \max\{0, \mathcal{Z}(p_2, q)\} \,| \, d\mu(q) = \\
&\mu(L_1(x_0)) \left(\| \mathcal{D}(p_1) - \mathcal{D}(p_2) \| + \| \mathcal{S}^{\text{cons}}(p_1) - \mathcal{S}^{\text{cons}}(p_2) \| + \| \mathcal{S}^{\text{prod}}(p_1) - \mathcal{S}^{\text{prod}}(p_2) \| \right).
\end{aligned}$$

Since $\mathcal{D}$ and $\mathcal{S}$ are continuous on Domain, it follows that $\mathcal{F}_0$ is continuous on Domain. □

Proposition 2.10 *The function $\mathcal{F} : C^* \setminus \{0\} \to C^*$ is continuous.*

Proof
The function $q \mapsto \eta(\mathcal{Z}(q, p_0))$ is continuous on Domain, and $\mathcal{F}_0$ is continuous on Domain, so the function $\mathcal{F}$ is continuous on Domain. Remains to prove the continuity of $\mathcal{F}$ on $C^* \setminus (\text{Domain} \cup \{0\})$. By definition, $\mathcal{F}(p) = p_0$ for all $p \in C^* \setminus$ Domain, so we only have to consider a sequence $(p_n)_{n \in I\!N}$ in Domain with limit $p \notin \text{Domain} \cup \{0\}$. Now, suppose the sequence $(\mathcal{F}(p_n))_{n \in I\!N}$ does not converge to p_0. Taking a subsequence if necessary, we may assume $\mathcal{F}(p_n) \neq p_0$, for all $n \in I\!N$. Note that $p \notin$ Domain means either $p \in \partial C^*$ or $p \in \text{int}(C^*) \setminus$ Domain.
In the former case, Corollary 2.4 and Lemma A.1.(1) imply $\liminf\limits_{n\to\infty}(\mathcal{V}(\mathcal{D}(p_n), p_0)) = \infty$.
In the latter case, Corollary 2.2 implies $\limsup\limits_{n\to\infty} \mathcal{G}(\mathcal{S}(p_n), p_0) = -\infty$.
Either way, we conclude

$$\liminf_{n\to\infty} \mathcal{Z}(p_n, p_0) = \liminf_{n\to\infty}(\mathcal{V}(\mathcal{D}(p_n), p_0) - \mathcal{G}(\mathcal{S}(p_n), p_0) - \mathcal{V}(w_{\text{total}}, p_0)) = \infty.$$

Hence, $\exists n_0 \in I\!N : \mathcal{Z}(p_{n_0}, p_0) \geq 1$. So, by (4) and (5), $\mathcal{F}(p_{n_0}) = p_0$. This is in contradiction with the assumption that $\mathcal{F}(p_n) \neq p_0$ for all $n \in I\!N$. □

This concludes the proof of the equilibrium existence theorem.

Appendix

A Supply and demand functions

In this appendix, we prove Lemma 2.1, Lemma 2.3 and Lemma 2.5.
Lemma A.2 and Proposition A.3 correspond with the first and second part of Lemma 2.1, respectively. The last part of Lemma 2.1 is proved in Corollary A.5.
The first part of Lemma 2.3 is a direct result of Lemma A.6 and Assumption 4.b. The continuity of the demand functions is proved in Lemma A.11. Lemma A.9 yields the last part of Lemma 2.3.
Finally, Walras' Law (Lemma 2.5) is a direct result of Lemma A.8.

Some results of [8], which we use in these proofs, are summarised in the following lemma.

Lemma A.1 *Let C be a salient half-space satisfying $V[C]$ is finite-dimensional and $C^{**} = C$.*

1. *Let S be a subset of C and let $p_0 \in \mathrm{int}(C^*)$. Then S is bounded if and only if the set $\{[x, p_0]_C \mid x \in S\}$ is bounded.*

2. *For all $p_0 \in int(C^*)$, the sets $K_1(p_0) := \{x \in C \mid [x, p_0]_C \leq 1\}$ and $L_1(p_0) := \{x \in C \mid [x, p_0]_C = 1\}$ are compact.*

3. *Let T be a closed set in C, satisfying $\forall x \in T : F_x \subset T$, let $p \in \mathrm{int}(C^*)$ satisfy $\mathcal{G}(x_0, p) = \sup_{x \in T} \mathcal{G}(x, p)$ for a unique $x_0 \in T$. Let $\alpha \in \mathbb{R}$. Then $K_\alpha^T(p) := \{x \in T \mid \mathcal{G}(x, p) \geq \alpha\}$ is compact.*

4. *Every $x_0 \in \mathrm{int}(C)$ is an order unit for C, i.e., $\forall x \in C\ \exists \lambda \geq 0 : x \leq_C \lambda x_0$. Moreover, for every sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathrm{int}(C)$ with limit $x_0 \in \mathrm{int}(C)$, there are sequences $(\psi_n)_{n \in \mathbb{N}}$ and $(\varphi_n)_{n \in \mathbb{N}}$ such that*

$$\psi_n x_0 \leq_C x_n \leq_C \varphi_n x_0 \text{ and } \lim_{n \to \infty} \psi_n = \lim_{n \to \infty} \varphi_n = 1.$$

Consider a production technology T_j, $j \in \{1, \dots, J\}$, with efficiency set $E(T_j)$. Without proof, we state that by Assumption 3.a, production technology T_j is a convex set in C.

Lemma A.2 *Let $p \in \mathrm{int}(C^*)$. Then the supply set $S_j(p)$ contains at most one element.*

Proof
Suppose both s_1 and $s_2 \in S_j(p)$ and $s_1 \neq s_2$. By Assumption 3.a, $s := \frac{1}{2}(s_1 + s_2) \in T_j \setminus E(T_j)$. Recall that for all $y \in C$ the set $F_y = \{x \in C \mid y^{\mathrm{prod}} \leq_{\mathrm{prod}} x^{\mathrm{prod}}$ and $x^{\mathrm{cons}} \leq_{\mathrm{cons}} y^{\mathrm{cons}}\}$. Since $T_j \setminus E(T_j) = \{x \in T_j \mid \exists y \in E(T_j), y \neq x : x \in F_y\}$, there exists $y \in E(T_j)$: $s \in F_y$. Now, since $p \in \mathrm{int}(C^*)$, $\mathcal{G}(y, p) > \mathcal{G}(s, p) = \mathcal{G}(s_1, p)$, which is in contradiction with s_1 being an element of the supply set $S_j(p)$. □

Proposition A.3 *The supply function $\mathcal{S}_j : Domain[j] \to E(T_j)$ is continuous.*

Proof
Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Domain$[j]$ with limit $p \in$ Domain$[j]$. Suppose the sequence $(\mathcal{S}_j(p_n))_{n\in\mathbb{N}}$ does not converge to $\mathcal{S}_j(p)$. Taking a subsequence if necessary, we may assume that

$$\exists \varepsilon > 0 \ \forall n \in \mathbb{N} : \| \mathcal{S}_j(p_n) - \mathcal{S}_j(p) \| \geq \varepsilon.$$

Define $x_n := \lambda_n \mathcal{S}_j(p_n) + (1-\lambda_n)\mathcal{S}_j(p)$ with $\lambda_n := \frac{\varepsilon}{\|\mathcal{S}_j(p_n)-\mathcal{S}_j(p)\|} \in (0,1]$, then, by Assumption 3.a, $x_n \in T_j \setminus E(T_j)$ and $\| x_n - \mathcal{S}_j(p) \| = \varepsilon$. The sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, so there is a convergent subsequence $(x_{nk})_{k\in\mathbb{N}}$ with limit $x \in T_j$ (Assumption 3.b), satisfying $\| x - \mathcal{S}_j(p) \| = \varepsilon$. Since $x_n = \lambda_n \mathcal{S}_j(p_n) + (1-\lambda_n)\mathcal{S}_j(p)$ with $\lambda \in (0,1]$, we find $\mathcal{G}(x_n, p_n) \geq \min\{\mathcal{G}(\mathcal{S}_j(p_n), p_n), \mathcal{G}(\mathcal{S}_j(p), p_n)\} = \mathcal{G}(\mathcal{S}_j(p), p_n)$. The function $\mathcal{G} : C \times C^* \to \mathbb{R}$ is continuous, so $\mathcal{G}(x,p) \geq \mathcal{G}(\mathcal{S}_j(p), p)$. Since $x \in T_j$, $x \neq \mathcal{S}_j(p)$, this is in contradiction with the properties of $\mathcal{S}_j(p)$. □

Corollary A.4 *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Domain$[j]$, with limit $p \in \mathrm{int}(C^*)$. If the sequence $(\mathcal{S}_j(p_n))_{n\in\mathbb{N}}$ is convergent with limit $s \in C$, then $p \in$ Domain$[j]$ and $s = \mathcal{S}_j(p)$.*

Proof
Since $\forall n \in \mathbb{N}\ \forall x \in T_j : \mathcal{G}(\mathcal{S}_j(p_n), p_n) \geq \mathcal{G}(x, p_n)$, the continuity of the function $\mathcal{G} : C \times C^* \to \mathbb{R}$ guarantees that $\forall x \in T_j : \mathcal{G}(s,p) \geq \mathcal{G}(x,p)$. By Assumption 3.b, the set T_j is closed, so $s \in T_j$ and Lemma A.2 yields $s = \mathcal{S}_j(p)$. □

Corollary A.5 *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Domain$[j]$ convergent to $p \in \mathrm{int}(C^*) \setminus$ Domain$[j]$. Then $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$, for any $p_0 \in \mathrm{int}(C^*)$.*

Proof
The sequence $(\mathcal{S}_j(p_n))_{n\in\mathbb{N}}$ does not have a point of accumulation, since existence of such a point would lead to a contradiction with the previous corollary.
Let $p_0 \in \mathrm{int}(C^*)$. By Lemma A.1.(3), for all $\alpha \in \mathbb{R}$ the set $L_{p_0}(\alpha) = \{x \in T \mid \mathcal{G}(x, p_0) \geq \alpha\}$ is compact, and so we find that $\forall \alpha \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n > N : \mathcal{G}(\mathcal{S}_j(p_n), p_0) \leq \alpha$. □

As the proof of Lemma 2.1 is complete herewith, we now concentrate on the proof of Lemma 2.3. Thereto, consider agent i, $i \in \{1, \ldots, I\}$, with the following characteristics: initial endowment $w_i \in C$, preference relation $\succeq_i$ defined on C, and shares θ_{ij} in the gain of production technology T_j, $j \in \{1, \ldots, J\}$. By the definition of the value $\mathcal{K}_i(p)$ on Domain we find

$$\forall p \in \text{Domain} : \mathcal{K}_i(p) = \mathcal{V}(w_i, p) + \sum_{j=1}^{J} \theta_{ij} \mathcal{G}(\mathcal{S}_j(p), p).$$

Note that $\mathcal{K}_i(p) \geq 0$, for all $p \in$ Domain. Since for every $j \in \{1, \ldots, J\}$ the supply function $\mathcal{S}_j$: Domain$[j] \to C$ is continuous, and since $\mathcal{G}$ and $\mathcal{V}$ are bi-continuous on $C \times C^*$, the value function $\mathcal{K}_i$: Domain $\to \mathbb{R}^+$ is continuous. Using $\mathcal{K}_i(p)$, the budget set $B_i(p) := \{x \in C \mid \mathcal{V}(x,p) \leq \mathcal{K}_i(p)\}$ is defined for every $p \in$ Domain, and therewith the demand set $D_i(p) := \{x \in B_i(p) \mid \forall y \in B_i(p) : x \succeq_i y\}$ consisting of all best elements of $B_i(p)$. Next, we shall derive some properties for this budget and demand set, using Assumption 4.

Lemma A.6 *Let $p \in$ Domain. Then the demand set $D_i(p)$ at pricing function p is non-empty.*

Proof

By Lemma A.1.(1), the budget set $B_i(p)$ is compact in C. For every $b \in B_i(p)$, define the set $G(b) := \{x \in B_i(p) \mid b \succ_i x\}$. The preference relation $\succeq_i$ is continuous (Assumption 4.c), so every set $G(b)$ is open. Suppose the demand set were empty, then every $b_0 \in B_i(p)$ is an element of at least one $G(b)$. The collection $\{G(b) \mid b \in B_i(p)\}$ is an open cover of the compact set $B_i(p)$, so there is a finite subset $F \subset B_i(p)$ such that $B_i(p) = \bigcup_{f \in F} G(f)$. The preference relation $\succeq_i$ being transitive, F has a maximal element $f_1 \in F$. Since, $f_1 \in G(f_2)$ for some $f_2 \in F$, $f_2 \neq f_1$, we arrive at a contradiction. $\Box$

As a direct result of the above lemma and Assumption 4.b, we can define the demand function $\mathcal{D}_i$: Domain $\to C$, where for every $p \in$ Domain, $\mathcal{D}_i(p)$ is the unique element of demand set $D_i(p)$. Before we prove the continuity of this demand function, let us state some preliminary lemmas concerning the budget set and the demand set of this agent.

Lemma A.7 *Let $p \in C^*$, let $w_i \in C$ satisfy $\mathcal{K}_i(p) > 0$, let $x \in C$, and suppose $x \succeq_i b$ for all $b \in B_i(p)$ satisfying $\mathcal{V}(b,p) < \mathcal{K}_i(p)$. Then $x \succeq_i b$ for all $b \in B_i(p)$.*

Proof

Let $b \in B_i(p)$ satisfy $\mathcal{V}(b,p) = \mathcal{K}_i(p)$. We shall prove that $x \succeq_i b$. Clearly, $b \neq 0$. So, for all $\tau \in [0,1)$ we have $\mathcal{V}(\tau b, p) < \mathcal{K}_i(p)$ and thus $x \succeq_i \tau b$. By Assumption 4.c, the preference relation $\succeq_i$ is continuous, so $x \succeq_i b$. $\Box$

Lemma A.8 *Let $p \in$ Domain. Then $\mathcal{V}(\mathcal{D}_i(p), p) = \mathcal{K}_i(p)$.*

Proof

In case $\mathcal{K}_i(p) = 0$, the budget set $B_i(p)$ equals $\{0\}$, and thus $\mathcal{V}(\mathcal{D}_i(p), p) = \mathcal{V}(0,p) = 0$. Now, suppose $\mathcal{K}_i(p) > 0$ and $\mathcal{V}(\mathcal{D}_i(p), p) < \mathcal{K}_i(p)$. Take $x_0 \in \text{int}(C)$ such that $x_0 >_C \mathcal{D}_i(p)$ and $\mathcal{V}(x_0, p) > \mathcal{K}_i(p)$ (cf. Lemma A.1.(4)). Consider the convex combination $\tau x_0 + (1-\tau)\mathcal{D}_i(p)$ with $\tau \in (0,1)$ so small that $\mathcal{V}(\tau x_0 + (1-\tau)\mathcal{D}_i(p), p) \leq \mathcal{K}_i(p)$. Then $\tau x_0 + (1-\tau)\mathcal{D}_i(p) \in B_i(p)$ and $\tau x_0 + (1-\tau)\mathcal{D}_i(p) >_C \mathcal{D}_i(p)$. By the monotony of preference relation $\succeq_i$ (Assumption 4.a), $\tau x_0 + (1-\tau)\mathcal{D}_i(p) \succeq_i \mathcal{D}_i(p)$. Since $x_0 \neq \mathcal{D}_i(p)$, we come to a contradiction with the uniqueness of the maximal element of $B_i(p)$, which is a direct result of Assumption 4.b. $\Box$

Lemma A.9 *Let $(p_n)_{n \in I\!N}$ be a convergent sequence in Domain with limit $p \in C^*$, and assume the sequence $(\mathcal{K}_i(p_n))_{n \in I\!N}$ is convergent with limit κ. If $\kappa > 0$ and the sequence $(\mathcal{D}_i(p_n))_{n \in I\!N}$ is bounded, then $p \in \text{int}(C^*)$.*

Proof

Let $\kappa > 0$ and let the sequence $(\mathcal{D}_i(p_n))_{n \in I\!N}$ be bounded. We may as well asume that the sequence $(\mathcal{D}_i(p_n))_{n \in I\!N}$ is convergent. Define $B_i(p, \kappa) := \{x \in C \mid \mathcal{V}(x,p) \leq \kappa\}$. Suppose $p \in \partial(C^*)$, then there is an element $x \in C \setminus \{0\}$, such that $\mathcal{V}(x,p) = 0$. Let $y \in B_i(p,\kappa)$, then by the monotony of $\succeq_i$ (Assumption 4.a), $y + x \succeq_i y$. By the strict convexity of $\succeq_i$

(Assumption 4.b), we find $y + \frac{1}{2}x \succ_i y$. Since $y + \frac{1}{2}x \in B_i(p, \kappa)$, we conclude that $B_i(p, \kappa)$ contains no maximal element with respect to preference relation $\succeq_i$. In order to arrive at a contradiction, we prove that the limit d of the sequence $(\mathcal{D}_i(p_n))_{n \in I\!N}$ is maximal in $B_i(p, \kappa)$. Indeed, let $b \in B_i(p, \kappa)$ satisfy $\mathcal{V}(b, p) < \kappa$. Then there is $N \in I\!N$ such that $\forall\, n > N : \mathcal{V}(b, p_n) < \mathcal{K}_i(p_n)$. So, $\mathcal{D}_i(p_n) \succeq_i b$ for all $n > N$. Continuity of the preference relation (Assumption 4.c) yields $d \succeq_i b$, and by Lemma A.7 we conclude that d is maximal in $B(p, \kappa)$. $\square$

To conclude this appendix, we prove that the demand function $\mathcal{D}_i$: Domain $\to C$ is continuous on its domain. For this we need the following lemma.

Lemma A.10 *Let $(p_n)_{n \in I\!N}$ be a sequence in Domain convergent to $p \in$ Domain. Then the following two properties hold.*

1) *If $b_n \in B_i(p_n)$ for each $n \in I\!N$, then there is a subsequence $(b_{nk})_{k \in I\!N}$ that converges to some $b \in B_i(p)$.*

2) *For each $b \in B_i(p)$ there exists a convergent sequence $(b_n)_{n \in I\!N}$ with limit b, such that $b_n \in B_i(p_n)$ for all $n \in I\!N$.*

Proof

1) Since $p \in \operatorname{int}(C^*)$ is an order unit, there is, by Lemma A.1.(4), a sequence $(\psi_n)_{n \in I\!N}$ in $I\!R$ satisfying $\lim_{n \to \infty} \psi_n = 1$, and

$$\forall\, n \in I\!N : \ \psi_n p \leq_{C^*} p_n.$$

Because $b_n \in B_i(p_n)$ for all $n \in I\!N$, we find $\psi_n [b_n, p]_C \leq [b_n, p_n]_C \leq \mathcal{K}_i(p_n)$. Since the function $\mathcal{K}_i$: Domain $\to I\!R^+$ is continuous, the sequence $(\mathcal{K}_i(p_n))_{n \in I\!N}$ is convergent. And since $p \in \operatorname{int}(C^*)$, by Lemma A.1.(1), boundedness of $[b_n, p]_C$ implies that the sequence $(b_n)_{n \in I\!N}$ is bounded. So, $(b_n)_{n \in I\!N}$ has a convergent subsequence $(b_{nk})_{k \in I\!N}$ with limit $b \in C$. Since $\forall k \in I\!N : \mathcal{V}(b_{nk}, p_{nk}) \leq \mathcal{K}_i(p_{nk})$, the limit b belongs to $B_i(p)$.

2) Let $b \in B_i(p)$. Since for all $p \in$ Domain : $0 \in B_i(p)$, we may as well assume $b \neq 0$. If $\mathcal{V}(b, p) < \mathcal{K}_i(p)$ then $\exists N \in I\!N \ \forall n > N : \mathcal{V}(b, p_n) < \mathcal{K}_i(p_n)$, and so, if we choose $b_n := b$ for all $n > N$, we are done. Therefore, we may as well assume $\mathcal{V}(b, p) = \mathcal{K}_i(p)$. For every $n \in I\!N$, define $\tau_n := \frac{\mathcal{K}_i(p_n)}{\mathcal{V}(b, p_n)}$. Note that $\lim_{n \to \infty} \tau_n = 1$. Now put $b_n := \tau_n b$, then $\forall n \in I\!N : \mathcal{V}(b_n, p_n) = \mathcal{K}_i(p_n)$ and $\lim_{n \to \infty} b_n = b$.

$\square$

Lemma A.10 expresses the type of continuity that we need in order to prove the continuity of the demand function $\mathcal{D}_i$.

Lemma A.11 *The demand function $\mathcal{D}_i$ is continuous on Domain.*

Proof

Suppose $\mathcal{D}_i$ is not continuous in $p \in$ Domain, then there is a sequence $(p_n)_{n \in I\!N}$ in Domain, converging to p, such that any subsequence of $(\mathcal{D}_i(p_n))_{n \in I\!N}$ does not converge to $\mathcal{D}_i(p)$. By

1) of the preceding lemma, the sequence $(\mathcal{D}_i(p_n))_{n\in I\!N}$ has a subsequence $(\mathcal{D}_i(p_{n_k}))_{k\in I\!N}$ that converges to some $b \in B_i(p)$. Now, the proof is done if we can show that $b = \mathcal{D}_i(p)$. Let $x \in B_i(p)$. By 2) of the preceding lemma, for all $n \in I\!N$ there is $x_n \in B_i(p_n)$ satisfying $x_n \to x$. Since the preference relation $\succeq_i$ is continuous (Assumption 4.c), we find that if $\forall n \in I\!N : \mathcal{D}_i(p_n) \succeq_i x_n$, then $b \succeq_i x$. So, $b = \mathcal{D}_i(p)$. □

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